The Extended Korteweg-de Vries Equation

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A slight and natural extension of the traditional Korteweg-de Vries equation (KdV) allows all (or groups) of its solitons to have the same velocity thus facilitating the application of the KdV to realistic quantum mechanical problems.

1. Introduction

The standard version of the famous Korteweg-de Vries equation (KdV) is given by

$$U_t(x,t) = v_0\, U_x + 6\, U\, U_x - M\, U_{xxx}\,, \qquad (1)$$

where v_0 stands for a constant velocity with which the system as a whole is translated in space. M represents the dispersion constant of the medium under consideration and the indices x and t denote partial differentiation with respect to these variables. Because of its nonlinearity (1) could not be solved completely until Gardener and collaborators [1] discovered a simple, yet ingenious Bäcklund transformation interrelating the KdV with the Sturm-Liouville or Schrödinger eigenvalue problem

$$H(x,t)\,\psi_n(x,t) = -M[\psi_n(x,t]_{xx} + U(x,t)\,\psi_n = E_n\,\psi_n\,. \qquad (2)$$

Since the solution of (1) appears formally as the potential of the Schrödinger equation (2), inverse scattering methods (i.e. via the Marchenko or Gelfand-Levitan equation) could be employed to follow the time evolution of $U(x,t)$ and to solve the KdV completely. Last not least, this lead to the analytical N-soliton solution, $U_N(x,t)$, of this equation,

$$U_N(x,t) = -2\,M[\ln(\det F)]_{xx}\,, \qquad (3)$$

$$F_{ij} = \delta_{ij} + \sqrt{f_i f_j}\cdot 2 \cdot (U_{0i}\,U_{0j})^{1/4}/(\sqrt{U_{0i}} + \sqrt{U_{0j}})\,, \qquad (4)$$

$$f_i = \exp(-\sqrt{2\,U_{0i}/M}\,(x - x_{0i} - h_i(t))\,, \qquad (5)$$

and to a better understanding of the concept of solitons in general. The constants x_{0i} in (5) denote the displacements the i solitons ($i = 1, 2, \ldots, N$) have at $t = 0$; the N by N matrix F of (3) is determined by (4) and (5), in which the symbol δ_{ij} stands as usual for the Kronecker-Delta and the U_{0i} are the amplitudes of the i solitons. To be specific, we give the one-soliton expression emerging from (3) to (5):

$$U_1(x,t) = U_1(x - h_1(t)) \qquad (6)$$
$$= -U_{01}\sec h^2(\sqrt{U_{01}/2\,M}\,(x - x_{01} - h_1(t))\,.$$

Equation (6) exhibits clearly the distinct features of the solitons of the KdV, namely that their amplitudes appear again in the arguments of their formfactors. A further point of note is that — in the case of the traditional KdV, (1), — the amplitude appears also in the function $h_i(t)$ determining the time evolution of the soliton, i.e.

$$h_i(t) = v_i t = 2\,U_{0i}\,t\,. \qquad (7)$$

Even more, the amplitudes of the asymptotic solitons (i.e. for solitons that are far enough separated so that they do not disturb each other) are in a unique way related to the energy eigenvalues, E_i, of the corresponding Schrödinger problem, (2):

$$U_{0i} = 2\,E_i = 2\,M\,K_i^2 = \hbar^2\,K_i^2/m \quad \text{with} \quad M := \hbar^2/2\,m\,. \qquad (8)$$

In this relation we gave an explicit definition of the dispersion constant M, which insinuates that we have some intention of considering (2) indeed as the quantum mechanical Schrödinger equation. But before we pursue this line of thought any further, we would like to reflect on relation (7). For water waves, for which the KdV was originally designed, such an amplitude-velocity dependence is quite natural. However, recently the KdV (and/or reflec-

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tionless potentials, i.e. KdV-solitons with $h_i(t)=0$) has also been applied to quantum mechanical problems in elementary particle [2], nuclear [3], and molecular [4] physics and in field theory [5]. The gist of these studies is that they follow the spirit of inverse methods, i.e. in their soliton approximations with $U(x,t) \cong U_N(x,t)$ this implies that the energy eigenvalues of the discrete part of the (experimentally accessable) spectrum of the Schrödinger equation (2), $\{E_i\}_{i=1,2,\ldots,N}$, serve as the only input for evaluating the potential, the wavefunctions and hence also the physical densities. The N-soliton solution of the KdV is thus interpreted as the potential of the quantum mechanical system under consideration and the KdV should be appropriate for describing its time evolution.

However, (7) and (8) determine the velocities of the N KdV-solitons in a highly undesirable manner: Different energy eigenvalues of the associated Schrödinger problem imply automatically different velocities of their related potential "lumps", i.e. KdV-solitons. If we now want to use the energy eigenvalues $\{E_i\}$ to construct for the time $t=0$ out of the N-soliton solution of the KdV the total potential $U(x,t)$ of the Schrödinger equation, then we have — due to the non-degeneracy of the one-dimensional Schrödinger problem — necessarily $E_i \neq E_j$ for $i \neq j$ (it is intended to use a pure single particle picture with one particle in one energy level), implying that all of the N solitons do have different velocities. Consequently the solitons, energy eigenvalues, potential-bags, or lumps or particles, are to move with different velocities. Hence, the whole "lump" does not remain together as required by the physics we set out to model. The obvious conclusion is that we may use the results of the inverse scattering theory (corresponding to KdV-solitons with the fixed time $t=0$), but that we can not rely on the KdV as a dynamical evolution equation. — Hence it has to be abandoned! Or to be modified.

Bearing these problems in mind, we would like to go anew through the standard procedure for deriving the KdV. Our aim is to find out whether it is possible to obtain a modification or extension of the KdV, which allows different solitons to have the same velocities. In doing so, we are not just interested in the mathematical structure of the problem, but also in physical applications and interpretations. This implies that we are not only dealing with number equations but with quantities that are characterized by a number and a dimension. To be specific, we give the potential, $U(x,t)$, and the energy eigenvalues, E_i, in units of MeV, the length, x, in fm, and the time, t, in seconds; i.e. U[MeV], x[fm], t[s] implying, according to (8), M[MeV · fm^2].

The equations given above are one-dimensional ones, but the substitutions $x \to r$ and $U(x,t) \to U(r,t) + M\,l(l+1)/r^2$ (the KdV, (1), represents the case with $l=0$; for $l \neq 0$ cf. e.g. (23) or [6]) together with the condition

$$U(x) = U(-x) \quad \text{or} \quad \psi_n(x=0)=0 \tag{9}$$

(which essentially fixes only the values of the x_{0i}; i.e. $x_{0i}=0$) facilitate the interpretation of (2) as the radial Schrödinger equation corresponding to three-dimensional spherically symmetric systems. Hence, the results based on (1) and (2) are not just valid in one dimension but they hold also for this restricted class of higher-dimensional problems. (Since the condition $x_{0i}=0$ implies some additional information on the related wavefunctions, the resulting potentials due to $U_N(x,t)$ are uniquely determined — provided $t=0$ is used — [7, 2, 3].)

In the following section it is shown how a slight modification of the traditional procedure leads to the extended KdV (EKdV). In Sect. 3 attention is drawn to the fact that similar extensions may also be given for the higher or generalized KdVs (corresponding to $l \neq 0$, cf. [6]) and to the modified KdV which is intimately related to (1) and (2). Section 4 contains a short summary.

2. Derivation of the Extended KdV and Discussion

First we go through some preliminaries to assess the possible time and velocity dependences of the wavefunctions ψ_n, cf. (2). To that end it is recollected that the physical content of (2) is that one gives up the hope of solving the full many-body Schrödinger equation, say $H(x,t)\,\Psi(x,t) = i\,\partial_t \Psi$, and that one makes the (physical) assumption that the interaction in H may be represented by an averaged mean field U generated by all N constituents of the system. (A compromise between this reduced problem and the desire to get a solvable system is to allow U to depend still on the time, thus retaining at least some reminiscence of the dynamics of the system.) Due to such a physical picture it is only natural to expect the mean field to couple the

different single-particle wavefunctions, ψ_n, with each other. Hence, it is not unreasonable to expect their arguments to be of the form $(\alpha_i - \pm K_i)$

$$\psi_n(x,t) = \psi_n\left(\sum_{i=1}^{N} \alpha_i (x - x_{0i} + v_0 t - v_i t)\right). \quad (10)$$

The symbols used have been defined in the introduction; e.g. the v_i are the velocities of the single-particle wavefunctions and hence also the ones of the associated potential contributions and densities. Introducing the constants $L_i = [\text{fm/MeV} \cdot \text{s}]$ we now rewrite expression (10) as

$$\psi_n(x,t) = \psi_n\left(\sum_{i=1}^{N} \alpha_i (x - x_{0i} + (v_0/L_0) L_0 t - 4\ E_i L_i t)\right), \quad (11)$$

where the E_i are as before the energy eigenvalues corresponding to the wavefunctions ψ_i. It is readily seen that the constants L_i and the generalized times τ_i[fm/MeV], defined by

$$\tau_i := L_i t\,, \quad (12)$$

are not that much characteristic for the system as a whole but rather for its i-th wavefunction (soliton). Hence, we refer from now on to τ_i as the eigentime of the i-th component/wavefunction of the system. It is the "time" with which the ψ_i evolve with the "velocity" $v_i = 4E_i$. Rewriting now $\psi_n(x,t)$ and the Schrödinger equation (2) in the forms

$$\psi_n(x,\tau) = \psi_n(x,t) = \psi_n\left(\sum_{i=1}^{N} \alpha_i (x - x_{0i} + (v_0/L_0)\tau_0 - 4E_i\tau_i\right) \quad (13)$$

and

$$H(x,\tau)\psi_n(x,\tau) = -M[\psi_n(x,\tau)]_{xx} + U(x,\tau)\psi_n = E_n\psi_n \quad (14)$$

we complete the preliminaries by defining the operator $\partial/\partial\tau$ in terms of the $\partial/\partial\tau_i$, i.e.

$$A := \partial_\tau A = \partial A/\partial\tau = \sum_{i=0}^{N} \partial A/\partial\tau_i\,. \quad (15)$$

To simplify the notation we identified $4E_0$ with v_0/L_0; $L_0 = 1$ fm/MeVs.

Now we continue in the traditional way [8], the only differences being that we use the (eigen) time τ instead of the usual t and that we are a bit more careful with the dimensions of the quantities involved. — The potential in (14) depends on τ but we want the E_n to be independent of this variable (or t) so that we demand as usual

$$\partial_\tau E_n = 0\,. \quad (16)$$

This condition for an isospectral flow of the Schrödinger problem is satisfied if there exists an unitary operator, T, connecting time-dependent Hamiltonian and wavefunctions with the time-independent one, i.e.

$$H(x,\tau) = T\,H(x,0)\,T^{+} \quad \text{and} \quad \psi_n(x,\tau) = T\,\psi_n(x,0) \quad (17)$$

or

$$U_\tau(x,\tau) = H_\tau(x,\tau) = [B, H(x,\tau)] \quad \text{and} \quad \psi_\tau(x,\tau) = B\,\psi_n(x,\tau)\,, \quad (18)$$

where the squared brackets denote the traditional quantum mechanical commutator and where the anti-Hermitian operator B remains to be specified. The first non-trivial choice for B is well known to be given by [1, 8]

$$B_3 = a_3 D^3 + a_1 D + a_0 = -4M D^3 + 3(UD + DU) + v_0 D/L_0 \quad (19)$$

(but for the L_0 which is required here to ensure the correct dimensions) resulting in the equation

$$U_\tau(x,\tau) = v_0 U_x/L_0 + 6\,U\,U_x - M\,U_{xxx}\,. \quad (20)$$

It is easily verified that (20) reduces for $L_i = 1$ fm/MeVs to the standard KdV of (1). To distinguish between the two versions we refer to the one with $L_i \neq 1$ as the extended KdV (EKdV).

The numerical values of the constants L_i are still at our disposition. With the appropriate choices, e.g. $L_i = v_p/4E_i$ $(i = 1, 2, \ldots, P < N)$ and $L_j = v_T/4E_j$ $(j = P+1, P+2, \ldots, N)$ they may be used to give groups (or all) of the N solitons different (or the same) velocities. In contrast to the traditional KdV they may move to the "right" or to the "left". In view of the fact that the characteristics (i.e. in particular their nonlinear superposition) of the N-soliton expression of the KdV, cf. (3) to (5), are obviously in no way influenced by the time dependence t or τ of $U_N(x,t)$; such a result could have been anticipated in advance. — In this sense our findings are naturally implicity already contained in older published work, yet, due to a bias induced by the application of the KdV to fluid dynamics, this is usually not recognized.

3. Extended and Higher KdVs and Extended Modified KdV

All the findings related to the extended KdV may be carried over to the higher or generalized KdVs. This can most easily be seen by taking recourse to the concept of hereditary symmetries and operators introduced by Fuchssteiner [9], who showed that the operator

$$\Phi = -M D^2 + 4U + 2U_x D^{-1} \quad \text{with}$$
$$D^{-1}((U_x)) := \int_0^x U(\xi)\,\mathrm{d}\xi \tag{21}$$

is hereditary. In here it suffices to recollect that this implies that repeated action of Φ on the right-hand-side of $U_\tau = U_x$, i.e.

$$U_\tau(x,\tau) = \Phi^i(U_x) \tag{22}$$

generates successively a sequence or family of equations, the descendants, all of which possess the same symmetries, conservation laws, etc. The appropriate hereditary operator giving "birth" to the KdV and its generalizations or higher forms is given by (21). It leads to

$$U_\tau(x,\tau) = \Phi^1 U_x = 6UU_x - MU_{xxx}, \tag{23}$$
$$U_\tau(x,\tau) = \Phi^2 U_x = M^2 U_{5x} - 10MUU_{xxx} - 20M U_x U_{xx} + 30U^2 U_x,$$
$$\vdots$$
$$U_\tau(x,\tau) = \Phi^n U_x = \ldots\ldots\ldots\ldots,$$
$$\vdots$$

which are readily checked to be indeed the (E)KdV and its generalizations. (To keep closer to the expressions commonly given in literature, we used $v_0 = 0$; usually one takes also $M = 1$ and the operator $\partial_t = \partial/\partial t$ instead of ∂_τ.)

It is not problematic to check that the whole procedure of Section 2. may be extended to any other (nonlinear) evolution equation. But without going through its derivation or its interrelation with the traditional version of the modified KdV, its extended version, the EmKdV is given:

$$V_\tau(x,\tau) = v_0 V_x/L_0 + 6V^2 V_x - MV_{xxx}. \tag{24}$$

The reason for doing so is motivated by the fact that it is the Miura (Bäcklund) transformation referred to at the very beginning, i.e.

$$U = V^2 + \sqrt{M}\,V_x \quad \text{with} \quad V =: \sqrt{M}\,[\psi_n]_x/\psi_n, \tag{25}$$

which helped Gardener et al. [1] to establish for the first time the famous interrelation of the KdV with the Schrödinger equation. — Recollecting that, according to the definition of Wigner's R-matrix, the logarithmic derivative of the wavefunction is expected to yield information on the resonances of quantum mechanical systems this extended version of the modified KdV is far from being of purely academic interest.

4. Summary

Interpreting the (linear) scattering problem related to the KdV as the quantum mechanical Schrödinger equation — instead of viewing it as the classical Sturm-Liouville equation — the derivation of the KdV is revisited. Dimensional considerations and a change of emphasis from the derivative in respect to the time t to the characteristic generalized eigentime τ, $\partial_\tau := \sum \partial\tau_i$ with $i = 0, 1, \ldots, N$, are shown to lead to an extended version of the KdV, the EKdV. The EKdV retains the attractive features of the conventional KdV and its soliton solutions (the characteristics of which may be taken from (3) to (5)). Yet, via the constants L_i (cf. Sect. 2), the numerical values of which are still at our disposition, it allows for more general time dependences than its literature version, (1).

Mathematically speaking, the extensions are of a rather trivial nature (unless the use of the operator ∂_t is insisted on, which would require recourse to an anti-Hermitian time-dependent operator B, cf. (18)). However, it is only these extensions which make it possible to introduce the EKdV and especially its solitons as a new and highly economic tool into nonrelativistic quantum mechanics.

It has been demonstrated [2] that the use of the inverse scattering method allows to circumvent the specific problems associated with the KdV-solitons. But since this procedure has to dispose completely of the dynamics of the system (it corresponds to the use of the N-soliton expression with the fixed time $t = 0$), the derivation of the EKdV presented above does indeed imply a distinct achievement facilitating further applications of the EKdV to quantum mechanics (and possibly stimulating

similar considerations for other nonlinear evolution equations):

The coupling of Schrödinger equation and EKdV leads to a physically sensible closed self-consistent system providing (hopefully) the basis for the construction of a realistic nonrelativistic field theory. Preliminary qualitative and quantitative results [2—5] support this notion which originally motivated the above derivation of the EKdV.

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Note added in proof: Let us denote the dispersion relation of the standard KdV, (1), by $\omega_{\mathrm{KdV}} = \omega_{\mathrm{KdV}}(k)$. (It determines the structure of the linearized version of the KdV uniquely.) In the case of a single soliton of the EKdV its resp. counterpart reads $\omega_{\mathrm{EKdV}} = L_1 \omega_{\mathrm{KdV}}$. For N solitons of the EKdV we get consequently $\omega_{\mathrm{EKdV}} = \sum_{i=1}^{N} L_i \omega_{\mathrm{KdV}}$; i.e. the EKdV is no longer a single partial differential equation but rather a system of coupled equations the constituents/solitons of which evolve according to their individually weighted dispersion relations, $L_i \omega_{\mathrm{KdV}}$. In the context of conventional applications of the KdV such a feature seems to be very unreasonable. But in view of the peculiar state (i.e. energy and momentum) dependences of the (effective) mass and hence also of the dispersion constant M, (8), as observed in nuclear physics it apparently *does* make sense in the present context.

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